Magnetic symmetry in quasicrystals

1992 J. Phys.: Condens. Matter 4 5997

(http://iopscience.iop.org/0953-8984/4/27/017)

J. Phys.: Condens. Matter 4 (1992) 5997–6008. Printed in the UK

Magnetic symmetry in quasicrystals

K Rama Mohana Rao† and P Hemagiri Rao‡
† Department of Applied Mathematics, AUPG Centre, Nuzvid 521 201, Andhra Pradesh, India
‡ Department of Mathematics, Bapatla Engineering College, Bapatla 522 101, Andhra Pradesh, India

Received 25 October 1991, in final form 6 March 1992

Abstract. The seven pentagonal and the two icosahedral point groups that represent the symmetries of quasicrystals in two and three dimensions are expressed in terms of composition series. Utilizing these series, eleven magnetic variants induced by the nine generating point groups are derived, employing the concept of colour generators. Using a computed character to represent a magnetic property, the idea of factor groups contained in a composition series is explored to enumerate simultaneously the maximum number of non-vanishing and independent constants required to describe a chosen magnetic property for all the point groups involved in the composition series and such of the magnetic variants that may exist. The case of magnetoelectric polarizability is worked out in detail for two composition series. The piezomagnetic, pyromagnetic and magnetoelectric polarizability tensors for all twenty magnetic quasicrystal classes have been obtained and tabulated. The results of this study are briefly discussed.

1. Introduction

It is well known that the limitation to the crystallographic point groups is due to the fact that crystals are solids with periodically arranged units, i.e. with translational symmetry. Compatibility between the translational symmetry and point symmetry of a crystal enforced that only those rotations or rotational inversions about an angle φ are allowed for which $2\cos\varphi \in \{-2, -1, 0, 1, 2\}$ is held. This warrants φ to assume one of the values $0, 2\pi, 2\pi/2, 2\pi/3, 2\pi/4$ or $2\pi/6$ only, the consequence being the existence of the so called 32 crystallographic point groups and 230 space groups. However, when the requirement of translational symmetry and hence the restriction of n to a certain value only is omitted, there are molecules with fivefold rotations and rotational inversions, some of whose symmetries were identified and reported in Brandmüller and Clauss (1988b).

Although it was believed for some time that periodic crystals with pentagonal symmetry ($\varphi = 2\pi/5$) cannot exist, theoretical and experimental evidence (Levine and Steinhardt 1984, Schechtman *et al* 1984) confirmed that quasicrystals with such symmetry can and do exist. The theoretical work of Levine and Steinhardt (1984) on quasicrystals became of great interest to many solid state physicists. The idea of a crystal with periodic translational order was systematically extended by these researchers to quasicrystals with quasi-periodic order by replacing the translation order by a long-range bond orientational order (BOO) with icosahedral symmetry. Following the discovery of Schechtmanite ($Al_{86}Mn_{14}$) with icosahedral symmetry by Schechtman *et al* (1984), there has been a tremendous burst of theoretical and experimental research activity towards understanding these structures (Gratias and Michel 1986, Mackay 1987). This culminated in the detection of many alloys exhibiting icosahedral phases similar to Al_4Mn. Several models have been proposed describing the structure of the icosahedral alloys, otherwise known as quasicrystals—the multiple twinning of cubic cells (Pauling 1987), the icosahedral glass in which the icosahedral units are randomly packed in the same orientation (Stephens and Goldman 1986), the three-dimensional (3D) Penrose tilings (Ogawa 1986) etc.

The successful attempts of Dubost *et al* (1986) and Bartges *et al* (1987) to solidify Al_6CuLi_3 quasicrystals; the discovery by Sen Gupta *et al* (1988) of optically active, transparent rare-earth pyro germinate (RPG) quasicrystals: $R_2Ge_2O_7$ and thulium pyro germinate (TPG) with unique crystal-field potential $\overline{10}\,m2$ (D_{5h}) site symmetry are a few interesting recent additions to the class of quasicrystalline substances exhibiting pentagonal/icosahedral symmetry. In so far as the studies pertaining to the geometrical and physical properties of these materials are concerned, whereas Sasisekharan *et al* (1988) performed x-ray diffraction studies on Al_6CuLi_3 and confirmed its icosahedral symmetry, Sen Gupta *et al* (1988) calculated the crystal-field effects on the magnetic properties of the TPG and obtained expressions for magnetic susceptibility tensors under the crystal field of D_{5h} symmetry. Brandmüller and Clauss (1988a, b) have calculated the irreducible tensors of rank 1–4 (without intrinsic symmetries) for all the irreducible representations (IR) of the seven pentagonal point groups. These irreducible tensors are useful and necessary for interpreting the Raman and hyper-Raman scattering. Yi-Jian Jiang *et al* (1990) obtained the piezoelectric, elastic, photoelastic and Brillouin tensors for the seven pentagonal and two icosahedral point groups in two and three dimensions (2D and 3D).

A good amount of theoretical understanding had been gained even before any quasicrystal was discovered. The Penrose tiling patterns of Penrose (1974) played a prominent role in the development of the theory of quasicrystals. It has been established theoretically that a 1D quasicrystal can be obtained by projecting a strip of a 2D square onto a 1D space and a 3D quasicrystal (of the Levine–Steinhardt pattern) can be obtained by projecting a slice of 6D lattice onto a 3D space (Conway and Knowles 1986).

In this article we derive the 11 magnetic variants induced by the seven pentagonal point groups $5(C_5)$, $\overline{5}(S_{10})$, $\overline{10}(C_{5h})$, $\overline{10}\,m2(D_{5h})$, $52(D_5)$, $5m(C_{5v})$, $\overline{5}2m(D_{5d})$ and the two icosahedral point groups $235(I)$ and $(2/m)\,\overline{3}\,\overline{5}(I_h)$ and determine the piezomagnetic, pyromagnetic and magnetoelectric polarizability tensors for all 20 magnetic quasicrystal classes. The authors believe that such a study should lead to a better understanding of the quasicrystalline materials reported in the literature and should provide useful information towards the identification of new materials bearing the envisaged symmetry. This paper is organized as follows. In section 2, the seven composition series, which are just sufficient for generating the 11 magnetic variants, are constructed and tabulated. In section 3 we apply the results of section 2. The method employed to induce the magnetic variants from a considered composition series is briefly explained and illustrated in section 3 with the help of two series. The magnetic variants obtained are tabulated employing a new notation also developed by these authors. The enumeration of independent constants required to specify a chosen magnetic property by each one of the 20 magnetic quasicrystal classes is carried out in section 4 employing the group-theoretical method. The non-vanishing as well as the independent tensor coefficients obtained in

Table 1. Pentagonal and icosahedral point groups in terms of the chosen composition series and the distinct magnetic variants induced from them.

Serial no	Composition series	Distinct magnetic variants induced from each series
1	$\overline{10}\,m2 \supset 52 \supset 5 \supset 1$	$52'$; $\overline{10}'\,m'2$
2	$\overline{10}\,m2 \supset \overline{10} \supset 5 \supset 1$	$\overline{10}'$; $\overline{10}'\,m'2'$
3	$\overline{10}\,m2 \supset 5m \supset 5 \supset 1$	$5m'$; $\overline{10}'\,m2'$
4	$\bar{5}\,2m \supset 52 \supset 5 \supset 1$	$\bar{5}'\,2\,m'$
5	$\bar{5}\,2m \supset 5m \supset 5 \supset 1$	$\bar{5}'\,2'\,m$
6	$\bar{5}\,2m \supset \bar{5} \supset 5 \supset 1$	$\bar{5}'$; $\bar{5}\,2'\,m'$
7	$(2/m)\,\bar{3}\,\bar{5} \supset 235 \supset 1$	$(2/m')\,\bar{3}'\,\bar{5}'$

respect of all three magnetic properties considered are presented in tables 4–6 (see later). The physical significance involved in the method adopted herein for the derivation of the magnetic variants as well as for the enumeration of magnetic constants is provided in section 5. Finally a brief discussion of the results obtained in this work is presented in section 6.

2. Formulation of composition series

It is well known that a series of the form $G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_s = \{e\}$ where $G = G_0$ is a group of finite order, G_{i+1} is a maximal normal subgroup of G_i, $i = 1, 2, \ldots,$ and $(s-1)$ is called a composition series of the group G. Since G is a group of finite order, this process shall ultimately terminate with the group G_s containing $\{E\}$. Because a maximal normal subgroup of a group is not unique, we obtain several such composition series for the group G under consideration. However, the factor groups G_i/G_{i+1} are unique but for isomorphism, and they need not occur in the same order.

In what follows, the seven composition series, just sufficient for generating the 11 magnetic variants which are induced by the seven pentagonal point groups and the two icosahedral point groups, are constructed and tabulated (table 1). Hermann–Maüguin (international) notation is adopted for denoting the point groups in the various series. It can be observed that the seven pentagonal point groups are subgroups of either of the groups $\overline{10}\,m2$ or $\bar{5}\,2m$ and that the icosahedral point group 235 is a subgroup of $(2/m)\,\bar{3}\,\bar{5}$. Furthermore, the point group 235 is not solvable, in that it has no non-trivial normal subgroup. In this paper, we contend that the alternating representation of the factor group G_i/G_{i+1}, where G_{i+1} is a subgroup of index 2 to the generating point group G_i, engenders an alternating representation of G_i, which, in turn induces a magnetic variant of G_i. The novel method of inducing the 11 magnetic variants is discussed briefly in section 3.

3. The method of generating magnetic variants

When an ordinary symmetry operation is applied on an arrangement of atoms in a molecule or crystal, although the geometrical structure may be brought into coincidence with itself, it may so happen that the orientations of some or all of the atomic magnetic moments (spins) are changed. The anti-symmetry operation R_2 was introduced into the realm of crystal physics to account for the reversal of the spins and bring the geometrical structure together with the spins into complete coincidence with itself. The introduction of R_2 paved the way for the derivation of 58 magnetic point groups (Tavger and Zaitsev 1956, Hammermesh 1962, Koptsik 1966, etc). These 58 magnetic groups together with the 32 conventional generating crystallographic point groups constituted the 90 magnetic crystal classes.

The magnetic group G_i' associated with the generating pentagonal or icosahedral point group G_i is derived in this section from the maximal normal subgroup G_{i+1} of index 2 of the group G_i in the series considered. Thus G_i' is induced from G_{i+1} by considering the semi-direct product of G_{i+1} with the double colour group E; R_2g, where $g \in G_i$ is the generator that generates G_i from G_{i+1} and R_2 is a colour-changing operation associated with g to form the appropriate colour generator R_2g. However if the maximal normal subgroup G_{i+1} is of index greater than two, no magnetic group can be obtained for G_i from G_{i+1}. This is due to the fact that the order of g in such cases is not equal to two and hence the incompatibility between the orders of R_2 and g in the colour generator R_2g. Furthermore, if the generating element g is the same in two different composition series for the considered group G_i, then no new magnetic variant G_i' is obtained for G_i from the later series. The remaining magnetic variants for G_i, if any, shall be obtained by considering other composition series involving G_i with the maximal normal subgroup $\bar{G}_{i+1} \cong G_{i+1}$.

The elegant method employed for the construction of the 11 magnetic variants induced by the nine generating point groups is exemplified now, with the help of two series—each involving pentagonal point groups and icosahedral point groups:

$$\overline{10}\,m2 \supset 52 \supset 5 \supset 1 \tag{1}$$

$$(2/m)\,\bar{3}\,\bar{5} \supset 235 \supset 1. \tag{7}$$

Consider the series (1). Following the observation made earlier with regard to the incompatibility between the orders of g and R_2, it can be seen that no magnetic variant can be generated from groups 1 to 5. The pentagonal point group 52 is generated from 5 by C_2'. Hence the appropriate colour generator is R_2C_2', where $R_2^2 = E$. The double colour group generated by R_2C_2' is E; R_2C_2' and the magnetic group associated with 52 is obtained by taking $5 \wedge (E, R_2C_2') = E, 2C_5, 2C_5^2, 5R_2C_2'$. We denote this group by $52'$. The colour generator from 52 to $\overline{10}\,m2$ is $R_2\sigma_h$ and the double colour group generated by this $R_2\sigma_h$ is E; $R_2\sigma_h$. Thus $52 \wedge (E, R_2\sigma_h) = E, 2C_5, 2C_5^2, 5C_2', R_2\sigma_h, 2R_2C_5, 2R_2C_5^3, 5R_2\sigma_v$ gives the magnetic variant induced by $\overline{10}\,m2$, which we denote as $\overline{10}'m'2$. Thus from the considered series (1), we obtain two magnetic variants $52'$ and $\overline{10}'m'2$ and these are shown in column 3 of table 1.

Since many quasicrystals show a global icosahedral symmetry as observed from diffraction patterns, we shall take up the icosahedral case with the series (7). The 60-element icosahedral group 235 has no non-trivial normal subgroup and hence no subgroup of index 2. Following the remark made earlier, no magnetic variant can be induced for 235 from (1). However, since i generates $(2/m)\,\bar{3}\,\bar{5}$ from 235, the lone magnetic variant induced by $(2/m)\,\bar{3}\,\bar{5}$ is obtained as $235 \wedge (E, R_2i) = E, 12C_5, 12C_5^2, 20C_3, 15C_2, R_2I, 12R_2S_{10}, 12R_2S_{10}^3, 20R_2S_6, 15R_2\sigma$. We denote this group by $(2/m')\,\bar{3}'\,\bar{5}'$. The generation of the rest of the magnetic variants can be done in a similar way through identifying the generators and forming the appropriate colour generators. The variants obtained from the different series are shown in table 1. The actual elements constituting these magnetic groups are provided in table 2.

Table 2. Magnetic variants generated by the pentagonal and icosahedral point groups.

Serial no	Pentagonal/ icosahedral point group G (in Hermann–Maüguin notation)	Maximal normal subgroup	Magnetic variant (G') induced by G	Elements of G'
1	$\bar{5}$	5	$\bar{5}'$	E, C_5, C_5^2, C_5^3, C_5^4, R_2i, R_2S_{10} $R_2S_{10}^3$, $R_2S_{10}^7$, $R_2S_{10}^9$
2	$\overline{10}$	5	$\overline{10}'$	E, C_5, C_5^2, C_5^3, C_5^4, $R_2\sigma_h$, R_2S_5, $R_2S_5^7$, $R_2S_5^3$, $R_2S_5^9$
3	$\overline{10}\,m2$	$\overline{10}$	$\overline{10}\,m'\,2'$	E, $2C_5$, $2C_5^2$, $5R_2C_2$, σ_h, $2S_5$, $2S_5^3$, $5R_2\sigma_v$
4	$\overline{10}\,m2$	52	$\overline{10}'m'\,2$	E, $2C_5$, $2C_5^2$, $5C_2$, $R_2\sigma_h$, $2R_2S_5$, $2R_2S_5^3$, $5R_2\sigma_v$
5	$\overline{10}\,m2$	$5m$	$\overline{10}'\,m2'$	E, $2C_5$, $2C_5^2$, $5R_2C_2$, $R_2\sigma_h$, $2R_2S_5$, $2R_2S_5^3$, $5R_2\sigma_v$
6	52	5	$52'$	E, $2C_5$, $2C_5^2$, $5R_2C_2^1$
7	$5m$	5	$5m'$	E, $2C_5$, $2C_5^2$, $5R_2\sigma_v$
8	$\bar{5}\,2m$	$\bar{5}$	$\bar{5}\,2'\,m'$	E, $2C_5$, $2C_5^2$, $5R_2C_2$, i, $2S_{10}^3$, $5R_2\sigma_d$
9	$\bar{5}\,2m$	52	$\bar{5}'\,2m'$	E, $2C_5$, $2C_5^2$, $5C_2$, R_2i, $2R_2S_{10}^3$, $2R_2S_{10}$, $5R_2\sigma_d$
10	$\bar{5}\,2m$	$5m$	$\bar{5}'\,2'\,m$	E, $2C_5$, $2C_5^2$, $5R_2C_2$, R_2i, $2R_2S_{10}^3$, $2R_2S_{10}$, $5\sigma_d$
11	$(2/m)\bar{3}\,\bar{5}$	235	$(2/m')\bar{3}'\bar{5}$	E, $12C_5$, $12C_5^2$, $20C_3$, $15C_2$, R_2I, $12R_2S_{10}$, $12R_2S_{10}^3$, $20R_2S_6$, $15R_2\sigma$

4. Enumeration of magnetic constants

The development of the concept of quasicrystals has challenged many widely held assumptions of crystallography and solid state physics. The discovery of icosahedral phases in metallic alloys has compelled theorists to reconsider several assumptions and to confront many new problems. Thus quasicrystals present a fundamental challenge to theoretical physicists to re-examine traditional concepts and devise methods for determining their physical properties. In this section we obtain the number of independent magnetic constants (n_i) needed by the 20 magnetic quasicrystal classes, in respect of the three known magnetic properties: (i) piezomagnetism, (ii) pyromagnetism and (iii) magnetoelectric polarizability.

A physical property is referred to as a magnetic property if either or both of the involved physical quantities is a magnetic field, magnetic induction or magnetic moment. Thus piezomagnetism is the appearance of magnetic moment $\boldsymbol{M}$ (M_i, $i = 1, 2, 3$) by the application of stress σ. Similarly, pyromagnetism is the appearance of magnetic moment $\boldsymbol{M}$ (M_i, $i = 1, 2, 3$) on the application of temperature t and magnetoelectric polarizability is the production of a magnetic field $\boldsymbol{H}$ (or $\boldsymbol{E}$) on the application of an electric field $\boldsymbol{E}$ (or $\boldsymbol{H}$) in a direction normal to it. The character $\chi(R_\varphi)$ corresponding to a symmetry element R_φ in the representation provided by each one of the three aforesaid magnetic properties

(Bhagavantam 1966) is:

$$\begin{aligned}\chi_a(R_\varphi) &= (4\cos^2\varphi \pm 2\cos\varphi)(1 \pm 2\cos\varphi)\\ \chi_b(R_\varphi) &= (1 \pm 2\cos\varphi)\\ \chi_c(R_\varphi) &= (1 \pm 2\cos\varphi)(\pm 1 + 2\cos\varphi).\end{aligned} \tag{4.1}$$

In equations (4.1), the + or − sign is to be chosen according to whether the symmetry operation R_φ under consideration is a pure rotation or rotation reflection through an angle φ.

The simple and elegant group-theoretic method, based on the concept of the factor groups contained in a composition series for obtaining simultaneously the number of independent constants (n_i) required to describe a chosen magnetic property by a generating point group and its magnetic variant (if any) is outlined hereunder, with the help of the IRs of the factor groups G_i/G_{i+1} contained in a composition series. The desired number of magnetic constants (n_i) is determined by utilizing the definition of the character of a coset† for the magnetic (physical) property considered and applying the known formula (Bhagavantam and Venkatarayudu 1951):

$$n_i = \frac{1}{N}\sum_\rho h_\rho \chi_\rho^{(\Gamma_i)}(R)\chi_\rho^{(\Gamma)} \tag{4.2}$$

with the usual notation. The case of magnetoelectric polarizability is illustrated hereunder for the two considered series in section 3.

It can be seen that the point group 1 requires nine magnetoelectric polarizability constants. From the point of view of the cosets, the point groups 5, 52 and $\overline{10}\,m2$, in terms of the factor groups 5/1, 52/5 and $\overline{10}\,m2/52$, can be expressed as

$$\begin{aligned} 5 &= 1 \cup C_5 1 \cup C_5^2 1 \cup C_5^3 1 \cup C_5^4 1\\ 52 &= 5 \cup C_2' 5\\ \overline{10}\,m2 &= 52 \cup \sigma_v 52.\end{aligned} \tag{4.3}$$

The character table of the factor group $5/1 \cong 5$ is:

5/1	E	C_5	C_5^2	C_5^3	C_5^4	n_i
A	1	1	1	1	1	3
^{1}E′	1	ω	ω^2	ω^3	ω^4	
^{2}E′	1	ω^4	ω^3	ω^2	ω	
^{1}E″	1	ω^2	ω^4	ω	ω^3	
^{2}E″	1	ω^3	ω	ω^4	ω^2	
$\chi_\rho^{(\Gamma)}$	9	$\frac{3+\sqrt{5}}{2}$	$\frac{3-\sqrt{5}}{2}$	$\frac{3-\sqrt{5}}{2}$	$\frac{3+\sqrt{5}}{2}$	

† For any magnetic or physical property, the character of a coset A_iH, $A_i \in (G\backslash H)$ in the factor group G/H is defined as the algebraic sum of the characters of all the elements contained in that coset in respect of that magnetic (physical) property divided by the order of that coset.

The factor group $5/1$ does not contain a 1D alternating IR. As such, 5 does not induce a magnetic variant. Putting $h_\rho = 1$, $\chi_\rho^{(\Gamma_i)} = 1\ \forall\rho$ and substituting the values of $\chi_\rho^{(\Gamma)}$ in (4.2), we find that group 5 requires three magnetoelectric polarizability constants.

The character table of the factor group $52/5$ is:

$52/5$	E5	$C_2'5$	n_i'
A′	1	1	2
B′	1	−1	1
$\chi_\rho^{(\Gamma)}$	3	1	

The coset $C_2'5$ consists of one conjugacy class in 52. The character of this coset for magnetoelectric polarizability, following the definition of the character of a coset, is 1. Since the alternating IR B′ of $52/5$ induces 52′ of 52, we find that the pentagonal group 52 and its magnetic variant 52′ require respectively two and one magnetoelectric polarizability constants.

Following the last of the equations (4.3), the character table of $\overline{10}\,m2/52$ can be written as

$\overline{10}\,m2/52$	52	σ_h 52	n_i''
A″	1	1	0
B″	1	−1	2
$\chi_\rho^{(\Gamma)}$	2	−2	

From the above table, by a reasoning similar to that given earlier, we find that the pentagonal class $\overline{10}\,m2$ and its magnetic variant $\overline{10}'m'2$ require zero and two magnetoelectric polarizability constants respectively. Thus the point groups 5, 52 and $\overline{10}\,m2$ contained in the series (1) require respectively three, two and zero constants, whereas the associated magnetic variants 52′ and $\overline{10}'m'2$ need one and two constants respectively to describe the magnetoelectric polarizability behaviour.

The icosahedral case is now illustrated with the help of the series (7). The point group 235 is a 60-element simple group with five conjugacy classes. The character table of the group $235/1 \cong A_5$, the alternating group on five symbols, is:

$235/1$	E	$15C_2$	$20C_3$	$12C_5$	$12C_5^2$	n_i
A	1	1	1	1	1	1
T_1	3	−1	0	τ	$-\tau^{-1}$	
T_2	3	−1	0	$-\tau^{-1}$	τ	
G	4	0	1	−1	−1	
H	5	1	−1	0	0	
$\chi_\rho^{(\Gamma)}$	9	1	0	$\dfrac{3+\sqrt{5}}{2}$	$\dfrac{3-\sqrt{5}}{2}$	

From the above table one finds that the group 235 requires only one magnetoelectric polarizability constant.

Since $(2/m)\bar{3}\bar{5} = 235 \cup i\,235$, the character table of $(2/m)\bar{3}\bar{5}/235$ becomes:

$(2/m)\bar{3}\bar{5}/235$	235	$i\,235$	n_i'
A′	1	1	0
B′	1	−1	1
$\chi_p^{(\Gamma)}$	1	−1	

From this table it follows that the group $(2/m)\bar{3}\bar{5}$ and its magnetic variant $(2/m')\bar{3}'\bar{5}'$ need zero and one constants respectively for the description of magnetoelectric polarizability.

Results obtained for the rest of the composition series of table 1 for the illustrated magnetic property and for all the series in respect of the other two magnetic properties are presented in table 3. The non-vanishing and independent tensor components in respect of each one of the twenty magnetic classes and for all the three considered magnetic properties are identified following the method of Bhagavantam (1966) and Nye (1985). These tensor components are calculated by solving the simultaneous equations, which arise when imposing the condition that the tensors are invariant under the elements of these respective magnetic classes. For the sake of brevity, we omit the somewhat lengthy calculations and present only the final results in tables 4, 5 and 6.

Table 3. The number of independent constants (n_i) required to describe the three magnetic properties of the 20 magnetic quasicrystal classes.

No	Pentagonal/ icosahedral point group G	Magnetic variant G' induced (if any) by G	Number of magnetic constants required: Piezomagnetism	Pyromagnetism	Magnetoelectric polarizability
1	5	—	4	1	3
2	$\bar{5}$		4	1	0
3		$\bar{5}'$	0	0	3
4	$\overline{10}$		4	1	0
5		$\overline{10}'$	0	0	3
6	$\overline{10}\,m2$		1	0	0
7		$\overline{10}\,m'2'$	3	1	0
8		$\overline{10}'m'2$	0	0	2
9		$\overline{10}'\,m2'$	0	0	1
10	52		1	0	2
11		$52'$	3	1	1
12	$5m$		1	0	1
13		$5m'$	3	1	2
14	$\bar{5}\,2m$		1	0	0
15		$\bar{5}\,2'm'$	3	1	0
16		$\bar{5}'2\,m'$	0	0	2
17		$\bar{5}'\,2'\,m$	0	0	1
18	235		0	0	1
19	$(2/m)\bar{3}\bar{5}$		0	0	0
20		$(2/m)\bar{3}'\bar{5}'$	0	0	1

Table 4. Piezomagnetic tensors for the 20 magnetic quasicrystal classes.

Magnetic quasicrystal class	Piezomagnetic tensor						
	0	0	0	Q_{14}	Q_{15}	0	
$5, \bar{5}, \overline{10}$	0	0	0	Q_{15}	$-Q_{14}$	0	(4)
	Q_{31}	Q_{31}	Q_{33}	0	0	0	
	0	0	0	Q_{14}	0	0	
$52, 5m, \overline{10}\,m2, \bar{5}\,2m$	0	0	0	0	$-Q_{14}$	0	(1)
	0	0	0	0	0	0	
$52', 5m', 52'm',$	0	0	0	0	Q_{15}	0	
$\overline{10}\,m'2'$	0	0	0	Q_{15}	0	0	(3)
	Q_{31}	Q_{31}	Q_{33}	0	0	0	
$\overline{10}'\,m'2, \bar{5}'\,2m',$	0	0	0	0	0	0	
$\overline{10}'\,m2', \bar{5}'2'm,$	0	0	0	0	0	0	(0)
$\overline{10}', \bar{5}', 235, (2/m)\bar{3}\,\bar{5},$	0	0	0	0	0	0	
$(2/m')\bar{3}'\bar{5}'$							

Table 5. Pyromagnetic vectors for the 20 magnetic quasicrystal classes.

Magnetic quasicrystal class	Pyromagnetic vector	
$5, \bar{5}, \overline{10}, 52', 5m', \bar{5}\,2'm', \overline{10}\,m'2'$	q_3	(1)
$52, 5m, \overline{10}\,m2, \bar{5}\,2m, \bar{5}'\,2m',$ $\overline{10}'\,m2', \overline{10}'\,m'2, \bar{5}'\,2'm, \overline{10}', \bar{5}'$ $235, (2/m)\,\bar{3}\,\bar{5}, (2/m')\,\bar{3}'\,\bar{5}'$	0	(0)

Table 6. Magnetoelectric polarizability tensors for the 20 magnetic quasicrystal classes.

Magnetic quasicrystal class	Magnetoelectric polarizability tensor			
	λ_{11}	λ_{12}	0	
$5, \bar{5}', \overline{10}'$	$-\lambda_{12}$	λ_{11}	0	(3)
	0	0	λ_{33}	
$52, 5m', \overline{10}'\,m'2,$	λ_{11}	0	0	
$\bar{5}'\,2m'$	0	λ_{11}	0	(2)
	0	0	λ_{33}	
$5m, 52', \overline{10}'\,m2',$	0	λ_{12}	0	
$\bar{5}'\,2'm$	λ_{12}	0	0	(1)
	0	0	0	
	λ_{11}	0	0	
$235, (2/m')\,\bar{3}'\,\bar{5}'$	0	λ_{11}	0	(1)
	0	0	λ_{11}	
$\bar{5}, \overline{10}, \overline{10}\,m2, \bar{5}\,2m,$	0	0	0	
$\overline{10}\,m'2', \bar{5}\,2'm', (2/m)\,\bar{3}\,\bar{5}$	0	0	0	(0)
	0	0	0	

5. Significance of the method

The derivation of magnetic symmetry groups and the enumeration of the number of independent magnetic constants employing the composition series technique has significance. This can be realized if the magnetic symmetry classes, generated here from various point groups G_{i+1} in the series, are associated with the appropriate alternating IR of the group G_i. In the little-group technique (Bhagavantam and Venkatarayudu 1951), it can be seen that, for the total symmetric IR Γ of the normal subgroup H (here the maximal normal subgroup G_{i+1}), the little group L (G_i, G_{i+1}, Γ) always coincides with G_i itself and the kernel (k) coincides with G_{i+1}—and thus $L/k \cong G_i/G_{i+1}$. Since the IRs of the factor group G_i/G_{i+1} engender those of the IRs of the same nature as G_i, the choice of G_{i+1} facilitates the calculation of the required IR of G_i to be engendered. For example, in the case of the point group $G = 52$, $G_{i+1} = 5$, the little group L $(52, 5, \mathrm{A}) = 52$ and $k = 5$. It can be seen that the alternating IR B' of the factor group $52/5$ engenders the alternating IR A_2 of 52. As $52'$ is generated from $G_{i+1} = 5$ in the composition series and since the alternating IR B' of $52/5$ in turn engenders the alternating IR A_2 of 52, the magnetic variant $52'$ may be associated with the IR A_2 of 52. A similar interpretation holds for the remaining ten magnetic variants generated in section 3.

It has already been established (Rama Mohana Rao 1987) that the number of independent constants required to describe a magnetic property and appearing before an IR μ of the factor group G/H is equal to the number of independent constants required by the corresponding coloured/uncoloured group of G induced by the IR λ of G, where the IR λ of G is engendered by the IR μ of G/H. Thus the process involved in the group-theoretic method, employed in this paper for enumerating the maximum number of non-vanishing and independent magnetic constants required for all the point groups involved in the composition series and such of the magnetic variants that may exist, has interesting physical significance.

6. Discussion

The idea of a colour generator, obtained by associating a double colour operation R_2 to the generator $g \in G_i$ from G_{i+1} to G_i in the series, where G_{i+1} is a subgroup of index 2 to G_i, is employed for the derivation of the magnetic variant induced by G_i. This method avoids considering each one of the generating point groups separately for the purpose of inducing its magnetic variants.

The nine generating point groups considered in this work are the quasicrystal symmetry groups in two and three dimensions. The number of non-vanishing and independent constants (n_i) required by the generating point group, together with such of the constants required by its induced magnetic variant, are obtained here for each one of the three magnetic properties considered, by considering the appropriate IRs of the factor group G_i/G_{i+1} in the series and by invoking the definition of the character of a coset and the formula (4.2). The advantage of the method lies in that the n_i needed for a chosen magnetic property for the generating point group G_i as well as its induced magnetic variant G_i' if any, can be obtained simultaneously; they need not be calculated separately.

The physical significance of the number of independent magnetic constants (n_i) appearing before the alternating IR of the factor group G_i/G_{i+1} emerges here, when the magnetic variant G' of G is regarded as being induced by the alternating IR of G/H. A similar interpretation can be extended to the other physical properties as well.

It is interesting to note that, in so far as the n_i are concerned, the 20 magnetic quasicrystalline classes divide themselves into different sets, with the classes contained in each set requiring the same n_i. Whereas for the piezomagnetism and magnetoelectric polarizability the 20 classes separate into four and five sets respectively, for pyromagnetism they separate into only two sets. Also, from tables 3 and 4, it can be seen that the quasicrystals whose symmetries belong to the nine magnetic classes $\bar{5}'$; $\overline{10}'$; $\overline{10}'m'2$, $\overline{10}'m2'$; $\bar{5}'2m'$, $\bar{5}'2'm$; 235; $(2/m)\bar{3}\bar{5}$, $(2/m')\bar{3}'\bar{5}'$ induced respectively by the six point groups $\bar{5}$, $\overline{10}$, $\overline{10}m2$, $\bar{5}2m$, 235, $(2/m)\bar{3}\bar{5}$ in each of which the centre of inversion (or reflection) has the character -1, do not require any piezomagnetic coefficients for their description. This is due to the fact that piezomagnetism is a centro-symmetric property and its coefficients are represented by axial tensors of odd rank.

The unique symmetries of quasicrystals play a central role in any discussion of their geometrical and physical properties. As such, group-theoretical methods underly much significance in the realm of quasicrystalline physics. The results presented in this paper are obviously theoretical. As such their utility in serving as valuable checks in the experimental determination of physical properties of any quasicrystal—we believe—will help further theoretical and laboratory studies in this relatively recent branch of science.

Acknowledgments

The authors are grateful to Professor L S R K Prasad, head of the Department of Applied Mathematics, AUPG Centre, Nuzvid for many illuminating discussions. They are thankful to the referees for their constructive suggestions regarding the manuscript.

References

Bartges C, Tosten M H, Howee P R and Ryba E R 1987 *J. Mater. Sci.* **22** 1663
Bhagavantam S 1966 *Crystal Symmetry and Physical Properties* (New York: Academic)
Bhagavantam S and Venkatarayudu T 1951 *Theory of Groups and its Application to Physical Problems* (Waltair: Andhra University Press)
Brandmüller J and Clauss R 1988a *Ind. J. Pure Appl. Phys.* **26** 60–7
—— 1988b *Croat. Chem. Acta* **61** 267–300
Conway J H and Knowles K M 1986 *J. Phys. A: Math. Gen.* **36** 45
Dubost B, Lang J M, Tanaka M, Sainfort P and Audia M 1986 *Nature* **324** 48
Gratias D and Michel L 1986 (ed) *Int. Workshop on Aperiodic Crystals (Les Houches, 1986) J. Physique Coll.* C3 47
Hammermesh M 1962 *Group Theory and its Application to Physical Problems* (Reading, MA: Addison-Wesley)
Jiang Yi-Jian, Liao Li-Ji, Chen Gang and Zhang Peng-Xiang 1990 *Acta Crystallogr.* A **46** 772–6
Koptsik V A 1966 *Shubnikovskie Gruppy* (Moscow: Moscow University Press)
Levine D and Steinhardt P J 1984 *Phys. Rev. Lett.* **53** 2477–9
Mackay A L 1987 *Int. J. Rapid Solid.* **2** 4
Nye J F 1985 *Physical Properties of Crystals* (Oxford: Clarendon)
Ogawa T 1986 *Bull. Japan. Inst. Met.* **25** 111
Pauling L 1987 *Phys. Rev. Lett.* **58** 365
Penrose R 1974 *Bull. Inst. Math. Appl.* **10** 266
Rama Mohana Rao K 1987 *J. Phys. A: Math. Gen.* **20** 47–57

Sasisekharan V, Baranidharan S, Gopal E S R, Sundaramurthy M and Sekhar J A 1988 *Pramana: J. Phys.* **30** L347–53
Schechtman D, Blech I, Gratias D and Cahn J W 1984 *Phys. Rev. Lett.* **53** 1951–3
Sen Gupta A, Battacharya S, Ghosh D and Wanklyn B M 1988 *Ind. J. Phys.* **62** 900
Stephens P W and Goldman A I 1986 *Phys. Rev. Lett.* **56** 1168
Tavger B A and Zaitsev V M 1956 *Sov. Phys.–JETP* **3** 430–7